5th Responsive Space Conference
RS5-2007-2003

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5th Responsive Space Conference
April 23–26, 2007
Los Angeles, CA

AIAA-RS5 2007-2003

SPACECRAFT SLEWING/GUIDANCE ALGORITHM FOR HYPERSPECTRAL IMAGERS

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Abstract:

Hyper Spectral Imagers (HSI) are planned as payloads on future Responsive Space missions. These payload instruments observe a scene spatially and, simultaneously, spectrally over a large number of discrete and contiguous spectral bands (in access of 200 bands) such that a complete spectrum is obtained for the region being imaged. The vast amount of data obtained is often referred to as a data cube. In order to make simultaneous measurements in both the spatial and spectral dimensions, scene images are constructed in push-broom fashion one multi-pixel line at a time. To obtain an accurate spatial/spectral image, the successive lines must be parallel and exposed for prescribed and comparable periods of time. An additional requirement for off nadir operation with scene swath paths (reasonably) free from the spacecraft footprint direction, results in highly complex spacecraft attitude motion. The quaternion based time histories of the motion attitudes and rates during the encounter can be determined, matched, and tracked. To maximize the number of targets visited during any engagement scenario, it is important minimize the time required to slew between targets. Since complicated spacecraft motions are present at the start of any one imaging episode, and to save settling and target acquisition time, it is desirable that the prescribed slew motions also incorporate, and prepare to match, the dynamical properties of the spacecraft motion of the upcoming scene dynamics; i.e., pointing direction and rate. A unique spacecraft slewing algorithm approach was developed that rapidly moves the spacecraft into the proper orientation and rate at the beginning of the imaging swath. The approach to slewing is quite general, versatile, adaptable, and applicable to a large variety of spacecraft and instrument types. This paper presents this slewing approach.

Introduction:

To fulfill the full promise of improved and more sophisticated components and subsystems, modern imaging satellites must actively and aggressively seek out targets situated in various directions, with lines of sight moving at various rates, and under stressing and marginal conditions. To operate under these conditions the satellite's systems must combine effective attitude and pointing control with mission planning and scheduling. A critical constituent in these operations is the kinematic maneuver of slewing the sensor system to match necessary attitudes and rates of the target. Indeed, improved slewing capability may be more important to overall mission efficiency than the fine pointing imaging requirements.

An individual imaging episode, whether visual or otherwise, usually takes a short time. The action of moving the line of sight (LOS) or changing the attitude to the proper kinematic condition often takes much longer and is the dominant action in the surveillance process. It is not unreasonable for slewing action to represent 70 to 80 percent of the whole process. Clearly then, even a small improvement in slewing capability translates, through time leveraging, to a larger improvement in imaging capability. More importantly, because slewing is to be as fast as reasonably possible, slewing taxes the kinematic and dynamic capabilities of the system much more than the "gentle" motions during the imaging interval itself. Indeed, slewing requirements dominate the design of the imaging system much more so than do the fine pointing requirements which, typically, reside in a sea of excess control capability.

This paper is concerned with approaches and solutions of the slewing problem for satellite based imaging systems. There are several types of slewing to consider. The first consists of a satellite with body fixed imager. In this case the entire spacecraft is rotated to the desired attitude in order to observe and follow the target. The outstanding example of this is, of course, the Hubble Space Telescope. A variation on the body fixed imager is to simply point the line of sight at the target without regard for the total attitude. A second type of slewing concerns a gimbaled type of telescope that is affixed to a large spacecraft base. This type of situation will occur if the satellite has several missions to perform and the telescope motion is to be decoupled from that of the base. Although the slewing problems are different for each case, they can be approached with some of the same basic concepts. This paper will consider the body fixed sensor while the gimbaled case is considered elsewhere.

Importance of Effective Slewing in Mission:

The following discussion of satellite operations is meant to highlight and justify the assertion concerning the importance of slewing. The primary focus of any mission is obtaining the necessary information and the design of the system seems to be focused on fine pointing and tracking. During the operational lifetime of the spacecraft only a very small portion of time is spent on the actual mission. Most of the time the spacecraft assumes a series of nominal attitudes that maintain readiness, collect and store electrical energy, and perform testing and calibration procedures. Of most interest during this phase are the calibration maneuvers which may point the sensor system at specific locations in space, uniform regions of the Moon, or certain broad regions of the Earth. In each of these cases, even though the pointing may have to be precise, the time to establish the test is quite generous and the slewing to the test attitude can be done at a relatively leisurely pace. As a result of this, slewing from the housekeeping and maintenance perspective can be casually designed; almost as an afterthought.

However, during the target encounter phase of the mission actions are more frenetic and the importance of slewing becomes paramount. To appreciate this recall that a satellite at an altitude of 400 km, for example, keeps a ground site within a 45 degree cone of nadir for less than 120 seconds. During this time span many targets may need to be observed and imaged. Imaging the target requires that not only must the pointing be accurate, but the spacecraft body rate must also precisely match the geometric line of sight rate. Analysis shows that under these conditions, slewing between targets can easily take 70%, or more, of the encounter time. In addition, to hasten slewing, the kinematics and dynamics function near their maximum capabilities. Slewing not only consumes the major portion of the encounter time, but also utilizes and stresses the maximum dynamic capabilities of the system. Consequently, proper and careful design and consideration of the slewing procedure can actually drive the design of the overall system.

Attitude Control Approach and Hierarchy:

The attitude control system operates in a cooperative environment with full knowledge of the past, present, and (near) future kinematic behavior between the satellite and target set. As a result, pointing direction and rate trajectories can be prepared beforehand and applied to the rigid body control system that then physically points the spacecraft. Figure #1 is a simple block

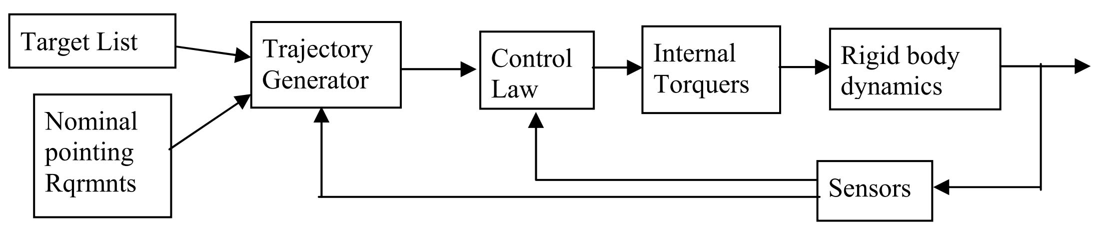


Figure 1: Attitude control law hierarchy

diagram of the situation.

The key component in the figure is the Trajectory Generator (TG) that creates a time history of the commanded attitude and body rate. The time history of the desired kinematic condition is fed to the body control system and the rigid body control system operates to track the generated trajectory. The key feature here is that the TG generates profiles that are compatible with the capabilities of physical control system. Thus, if after finishing an action it is desired to put the satellite into an LVLH attitude, for example, the TG will determine a smooth continuous trajectory that is fed to the control system and which the control system can follow. Likewise, when trying to observe a set of targets, TG will generate a trajectory profile between successive targets that is continuous, within dynamical constraints, and that the control system can follow. In this way, the design of the control system is decoupled from the generation of the trajectories. However, the trajectories are generated with the control system capabilities in mind. First design the best rigid body control system, then adapt the trajectory generator to these capabilities.

The slew that we perform is not that of tracking and converging on the current position and rate of the target, but, rather, of anticipating and intercepting the target at a future point in time. Since we have full knowledge of the encounter geometry we can plan ahead and generate a complete trajectory to the target and intercept the target and its motion. This is, typically, faster and more efficient than wasting time and energy following the target as if we didn't know where it was going or what it was going to do. The pointing and tracking problem we are performing here is different than the tracking and pointing problem that results from trying to follow a non-cooperative target.

Kinematic and Dynamic Notation Elements:

The rotational orientation, or attitude, of a rigid body with respect to a fixed/inertial reference frame can be indicated in a number of ways. Several of the most common, used here as needed, are: (1) rotation matrix, C (also called direction cosine matrix, DCM); quaternions, $\vec{Q}$; and Euler angles.

The rotation matrix is an orthonormal matrix that sets the "gold" standard for representation in that it uniquely defines the rigid body attitude and is easy to use in analysis and application. However, because the rotation matrix has nine components, some workers (uncritically) feel that it requires excessive numerical computation steps and prefer using quaternions. This is not necessarily so[12].

The quaternion requires only four (constrained) parameters to do the "work" of the nine component rotation matrix and is currently frequently used in analysis and applications. However, the quaternion requires care in application since it is not unique in that each body orientation is represented by exactly two quaternions. It should be pointed out that rotation matrices often accompany any kinematic notation used because they are used to transform between various other notations that may be required. We mostly use quaternions here and the debate between using quaternions or rotation matrices is not entered here.

The Euler angles, which are very appealing in some cases, are much more difficult to utilize in analysis and application and only mentioned here because they are the natural notation when the pointing system is mounted on gimbaled axes.

We denote a rotation quaternion by $\vec{Q} = [q_1, q_2, q_3, q_4] = [\hat{e}\sin(\theta/2), \cos(\theta/2)]$, where $\hat{e}$ is the unit vector about which the rotation takes place and θ is the rotation angle (as per Euler's rotation theorem). $\vec{Q}$ is not a vector in the usual sense. The composition of two rotations, defined by quaternions $\vec{Q}_1$ and $\vec{Q}_2$, is performed by the quaternion multiplication $\vec{Q}_3 = \vec{Q}_1 \otimes \vec{Q}_2$, where the operation $\otimes$ is defined in the usual manner[1,2,3,7].

The reader is assumed to be familiar with these methods.

More Complete Definition of Slewing and Slewing Statement:

Often, when discussing a specific slewing action, only the rotational angle to be slewed is mentioned. This description is inadequate and also misleading since it may imply that the LOS is at rest at both the start and end of the slew. For imaging Earth based targets from moderate altitude satellites this is far from the case because the LOS to the target can be moving rapidly; one degree per second (or greater) is typical. In order to specify the required slew, it is essential to indicate (in the least) both the attitudes and body rates at the slew's start and end. **This point cannot be stressed often enough.** For example, a simple slew through a small angle may appear to only require several seconds, but when the rates at both ends are considered, the slew time may be substantially longer and the angular excursion surprisingly larger. Indeed, under more demanding conditions higher order LOS rates may need to be considered and matched.

Slewing Visualization:

Helpful in visualizing slewing is the rotational sphere in Figure 2. The observer, or satellite, is at the center of the sphere, and the sphere's surface (unit distance) consists of all possible (4π steradian) LOS pointing directions. At the center is the inertial frame (X_I,Y_I,Z_I) against which all body orientations/attitudes are measured. Also shown at the center is the rotated body frame (X_b,Y_b,Z_b). The attitude of the rotated body frame is denoted by the quaternion, $\vec{Q}_A$, which is shown on the sphere surface. The unit vector, $\hat{R}$, is the LOS direction within the body and rotates with the body. For convenience, a copy of the frame (X_b,Y_b,Z_b) is translated onto the LOS tip of the rotation sphere to indicate the full attitude. The changing LOS vector has velocity vector $\vec{v}_n$, which is normalized to $\hat{R}$. Because of $\vec{v}_n$, the body has an instantaneous body rate vector $\vec{\omega}_A$ (whose line goes through the sphere center) and the circle about $\vec{\omega}_A$ scribes the tip of the LOS direction (if the body rate, $\vec{\omega}_A$, maintains a constant direction.) The circle is planar, perpendicular to $\vec{\omega}_A$, and, typically, not a great circle of the sphere. The combination of the LOS, projected body frame, and rate circle provide a useful way of depicting the initial attitude and rate information.

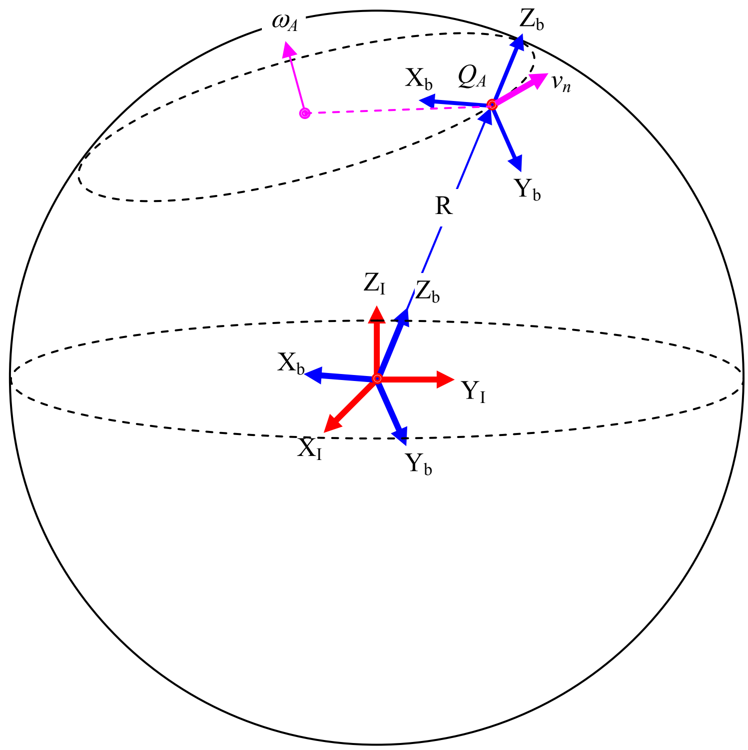


Figure 2: Directional sphere to view slewing

Statement of Slewing Problem and Initial Assessment of Difficulty:

Using the visualization tool just described, we state and discuss the basic slewing operation which forms the basis of all subsequent work. See Figure 3.

The objective is to:

(1) rotate a rigid body with an initial attitude quaternion and body rate, $(\vec{Q}_A, \vec{\omega}_A)$, to a final quaternion and body rate, $(\vec{Q}_B, \vec{\omega}_B)$, in minimum time, while,

(2) satisfying system constraints on maximum accelerations, rates, etc.

The initial orientation quaternion and body rate, $\left(\vec{Q}_A, \vec{\omega}_A\right)$, is indicated by the ◎ symbol on the upper circular slice. The final quaternion and body rate, $\left(\vec{Q}_B, \vec{\omega}_B\right)$, is indicated by the ☺ symbol laying on the canted circular slice near the bottom of the figure. As mentioned above, each circular slice need not be a great circle and each is tangent, at the start and end points, to the respective instantaneous velocities of the line of sight (LOS). The objective is to find a minimum time trajectory of the quaternion and corresponding body rate, $\left(\vec{Q}(t), \vec{\omega}(t)\right)$, that connects the two end points, matches the body rates at the two end points, and does not exceed other dynamic constraints during the course of the trajectory. Dynamic constraints include: acceleration constraint because of maximum torquer capabilities, rate limits because reaction wheel speed limits and possibly star tracker operational rate limits. Assuming that such a trajectory exists, we depict it by the red curve laying on the surface of the unit sphere. *REMARK: The problem of slewing between rest points where the rates are zero is frequently, and adequately, discussed in the literature. ENDREMARK*

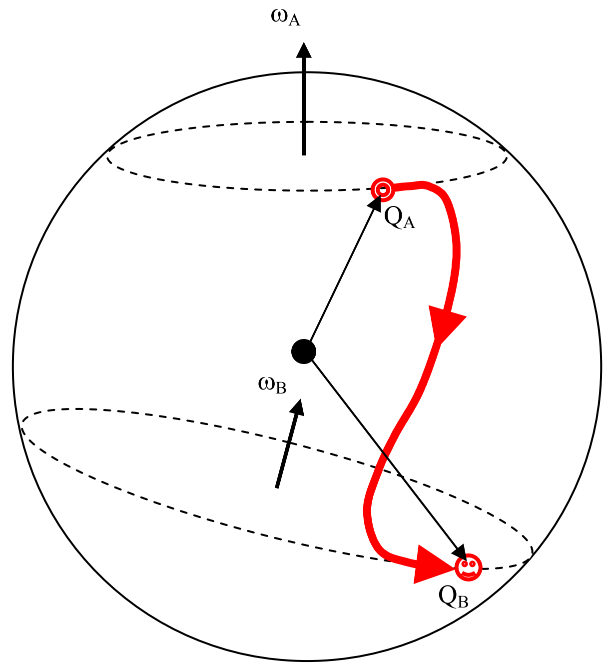


Figure 3: A potential minimum time slew.

The time optimal trajectory (red curve) is difficult to determine. Determining it requires not only knowledge and experience with encounter geometry, optimal control and dynamics[3], but also with mathematical topics such as differential geometry, curves on manifolds, and other mathematical disciplines. Robotics[8], with its need to track the location and pose (attitude) of linkages, joints, and end-effectors, addresses many of the topics considered here. In addition, the field of Computer Graphics is another useful source for results in this area and contains several excellent papers with interesting and applicable results[9,10,11,12,13]. The reader is cautioned that in the computer graphics field, rotation matrices are viewed from the perspective of fixed (inertial) frames and are inverses (transposes) of the same rotation matrix as viewed in the spacecraft field where the rotation matrix (usually) transforms vectors to the rotating body frame.

Justification for Simpler and Basic Approach:

The kinematics and dynamics of time optimal rotational slewing of a rigid body are complex and difficult to calculate[1,2,3,4,5,6]. For practical purposes we seek an alternate, plausible, and workable (i.e., computationally inexpensive) solution even if it is not time optimal. The solution we present is the Three Segment Slew (TSS); a solution that meets these objectives and exhibits flexibility that permits extension to other situations and the inclusion of further limits and constraints. Before presenting the TSS solution, it is worthwhile to point out some of the difficulty encountered in finding a minimum time slew that matches the quaternion and body rate end conditions. This will provide some justification for the intuitive and plausible approach for slewing proposed later. In order to address almost certain questions and objections to this approach, it is appropriate to consider a (seemingly) simple slewing problem[5,6] and to show how really difficult it is to determine a time optimal slew procedure.

We begin with a brief discussion of the problem of rotating an ideal sphere about one of its principal axes (all axes through the sphere center are principal axes). Figure 4 shows the sphere with a set of principal axes drawn as shown. The moment of inertia matrix is symmetric,

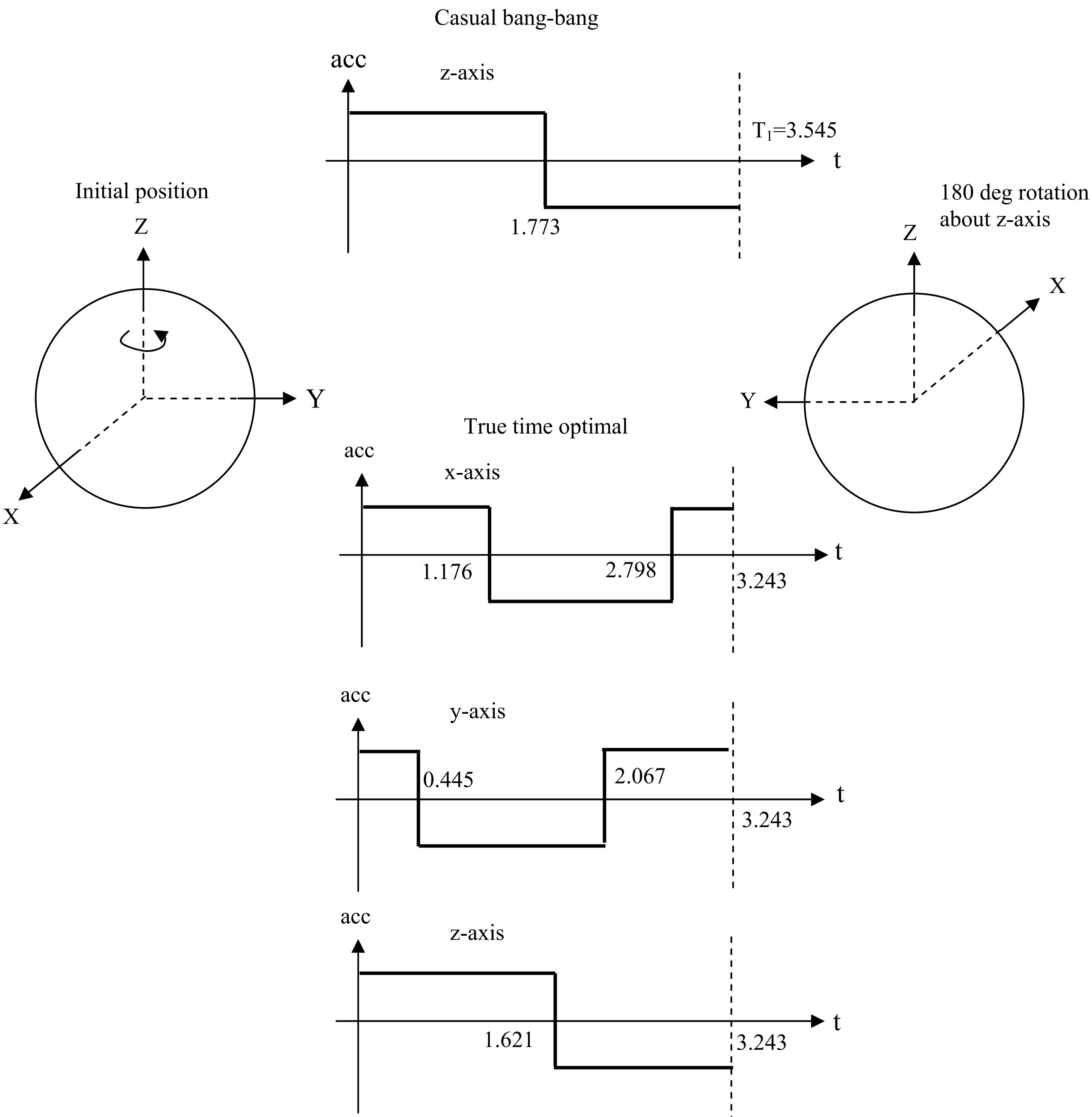


Figure 4: True time optimal control to rotate inertially symmetric body

diagonal, and has all three moments of inertia equal. (Note: although less obvious, a simple uniform cube has similar second order mass properties.) The problem is to rotate the sphere 180 degrees from rest to rest about the z-axis in minimal time. If the maximum torque that can be applied is T_{max} Nm, the maximum available acceleration is $\alpha_{\max} = T_{\max}/J$, and a casual application of the bang-bang principle determines the minimum slew time $t = 2\sqrt{\theta/\alpha_{\max}} = 2\sqrt{\pi/\alpha_{\max}} = 3.545\sqrt{\alpha_{\max}^{-1}}$. This torque profile is only applied about the z-axis and the other two axes take no action whatever. However, Bilimora & Wie[5] and Byers & Vadali[6], show that there is a solution that is faster by about 8%. Their solution, however, is much more complex, requires iterative search techniques, and is very difficult to determine. The true time optimal solution requires bang-bang torquing on all three axes, each with three reversals as shown in Figure 4. A physical interpretation of what is happening is that in addition to the single axis slew about the z-axis, one of the other two torques initiates a rotation about one of the other axes and the third torque acts perpendicular to it to elicit a precession motion. The additional procession motion helps speed up the total rotation. However, this is done at a tremendous cost; all three axes are torqued, and the total expenditure of power and energy is almost tripled (×2.76); just to get an 8% reduction in time. In addition to the extra energy required, the computational effort is huge and can only be appreciated by reviewing the references mentioned. This result is for a simple 180 degree rotation. For other rotations, in particular, less than about 72 degrees, additional reversals in torque are required.

The problem and solution just described is based on only the acceleration level being limited. If additional constraints on angular velocity and angular excursion are imposed, the solution becomes even more complex and not even mentioned in the referenced articles.

Enough said on this.

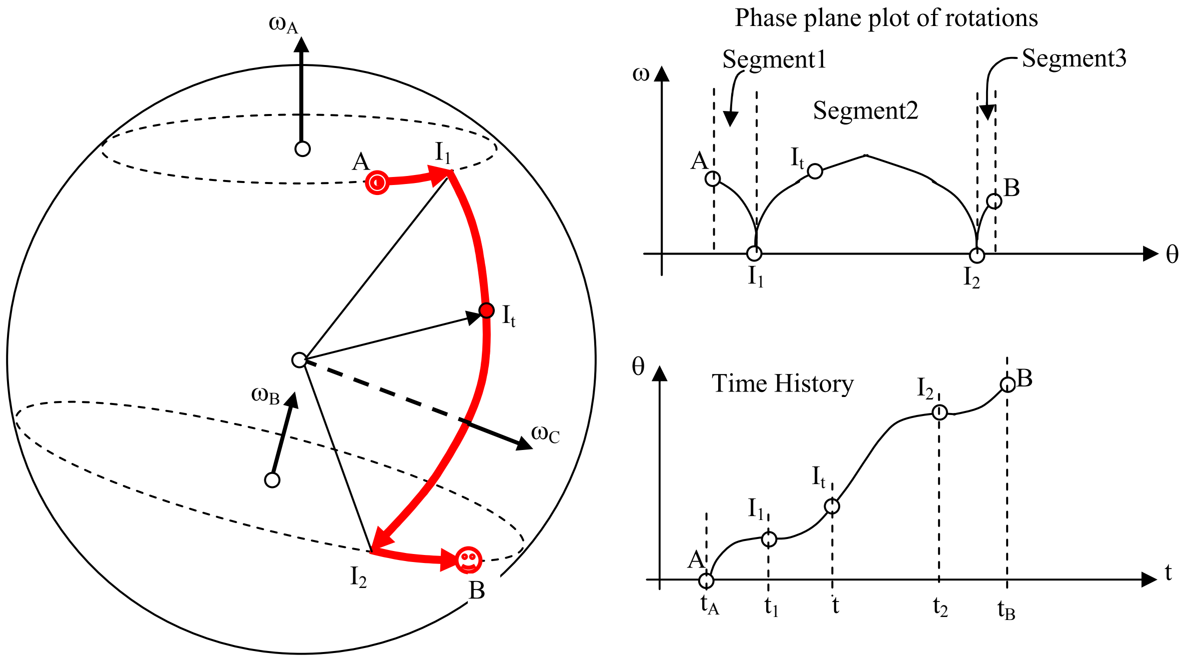
The point of the lengthy discussion above is that time optimal problems are very difficult, and not suited for on-board implementation. Our alternative is to strive for intuitive, plausible, and simple solutions; even at a cost of increased slew time.

With no further apologies, we present the Three Segment Slew (TSS); a solution that meets these objectives and exhibits flexibility that permits extension to other situations and the inclusion of further limits and constraints.

Three Segment Slew:

The basic rational for the Three Segment Slew is given in this section while specific equations and procedures for the acceleration limited case are detailed in Appendix A. The reader is encouraged to refer freely to appendix A while reading this section.

The Three Segment Slew is performed as shown in Figure #5 which shows the motion of the LOS in the rotation sphere, a phase plane plot of the time trajectory (in a folded form), and a time history (also in a folded form). Letting $A \Leftrightarrow (\vec{Q}_A, \vec{\omega}_A)$ and $B \Leftrightarrow (\vec{Q}_B, \vec{\omega}_B)$ represent the starting and ending position quaternions and body rates, the slew can be performed in three steps, each of which is a single axis rotation. First perform a maximum (negative) rotational acceleration about the axis, $\hat{e}_1$, collinear with $\vec{\omega}_A$, such that $\vec{\omega}_A$ is reduced to zero. This locates the first intermediate point $(\vec{Q}_{I1}, \vec{0})$, which is a rest point. Locate the second intermediate point $(\vec{Q}_{I2}, \vec{0})$ obtained from $(\vec{Q}_B, \vec{\omega}_B)$ by a backward maximum rotational acceleration about the rotation axis, $\vec{e}_3$, collinear with $\vec{\omega}_B$. The two intermediate points $(\vec{Q}_{I1}, \vec{0})$ and $(\vec{Q}_{I2}, \vec{0})$ are rest



Point A: $\{Q_A, \omega_A\}$; Segment1: $A \rightarrow I_1$
Point I_1: $\{Q_{I1}, 0\}$;
Point I_t: $\{Q_t, \omega(t)\}$: Segment2: $I_1 \rightarrow I_2$
Point I_2: $\{Q_{I2}, 0\}$
Point B: $\{Q_B, \omega_B\}$: Segment3: $I_2 \rightarrow B$

Figure 5: Basic Three Segment Slew showing rotation sphere, phase plane plot, and time plot

points and, therefore, a minimum time rest-to-rest eigenaxis slew[3] can be performed between them according to the rotation axis for the difference quaternion $\vec{Q}_e = \vec{Q}_{I1}^{-1} \otimes \vec{Q}_{I2}$. The complete operation consists of

(1) bringing the initial point $(\vec{Q}_A, \vec{\omega}_A)$ to rest at $(\vec{Q}_{I1}, \vec{0})$ using a single axis rotation about $\hat{e}_1$,

(2) performing an eigenaxis rest-to-rest slew between $(\vec{Q}_{I1}, \vec{0})$ and $(\vec{Q}_{I2}, \vec{0})$, using $\vec{Q}_e = \vec{Q}_{I1}^{-1} \otimes \vec{Q}_{I2}$ and then

(3) completing the slew trajectory with a single axis rotation about $\hat{e}_3$ from $(\vec{Q}_{I2}, \vec{0})$ to $(\vec{Q}_B, \vec{\omega}_B)$.

The procedure just described provides the trajectory path with no explicit association to time. The time it takes to perform each of the three steps above depends on the capabilities of the system and the restrictions and constraints involved. Appendix A details the acceleration limited case.

For example, if the system is acceleration limited, then each of the segments would (most likely) follow a bang-bang control law in order to move the trajectory between the segment end points as (nearly) quickly as possible. If there is a further rotational rate restriction, then the control law would include a zero acceleration coast portion when the maximum rate is achieved; i.e., a bang-coast-bang control profile.

Should it be necessary to limit dynamic jerk so that there is reduced possibility of exciting structural vibrations, then the profiles in each segment can be adjusted accordingly.

The main point here is that partitioning the trajectory into the three segments shown will rotate the body between the start and end points. Additional restrictions and constraints can be addressed to each of the segments independently of the others.

Finally, it should be pointed out that, although we strive for (near) minimum time trajectories, the Three Segment Slew trajectory is rarely time optimal. It will be time optimal in certain special cases, but generally, the two rest stops are expensive in terms of time. However, the rest stops provide the dynamics with important capabilities as discussed below.

Discussion of Three Segment Slew:

1) The TSS algorithm described above is not a minimum time solution in most situations.

2) The seemingly sharp corners at the intermediate points in the trajectory of Figure 5 do not represent violent and abrupt changes in motion. Rather, the corners occur at zero rate and at these points the trajectory can be made to move in any direction without penalty.

3) The algorithm determines both the time and trajectory between dynamic points *A* and *B*. Since *B* is moving, its location depends on the time required to get there. Thus, it is necessary to establish an Intercept Algorithm that determines the earliest time (using the TSS) that a moving target can be intercepted. It is straight forward to determine a suitable algorithm using the TSS as a basic component in the calculation.

4) The TSS algorithm is used only once before the start of the slew to the next target. Basically, the TSS determines a small number of parameters that define the trajectory. For example, in Figure 5, (see Appendix A) all that is necessary to determine the motion in each of the three segments is the rotation axis direction, the total angle to be rotated, and the maximum allowable accelerations and rates. With this simple set of data, the trajectory generator can generate the trajectory profile at any desired time step and provide it to the rigid body controller. It is not necessary to redo the TSS at every time step.

5) While the rest stops may appear to be wasteful of time, they provide a convenient method for extending slew times. In many problems it is more important to view the target at a specific time (because of more favorable viewing conditions) than in getting to it as quickly as possible. In such a situation it is very difficult to design a trajectory that not only satisfies the end points, but also requires a prescribed time. The rest points (actually, only one is needed) provide a place for the LOS to loiter awaiting the target. Were it not at rest, the LOS would keep moving, and intercepting the target would be difficult. But in this case, simply slew to the second rest point and wait there until the appropriate time and then accelerate to meet the target at the proper location and at the proper rate.

6) If additional constraints are required of the system, they can be implemented on each of the segments separately.

7) It was pointed out above that the TSS is not necessarily a minimum time solution. However, it is a valid solution. As such, it can be used as a starting trajectory in any of the iterative techniques used to optimize paths and curves in space. Methods of improving on the TSS solution should be considered if they can reduce slew time at a reasonable cost in computational effort.

8) As indicated later, the rest points provide suitable places to anchor Keep Out zone routes.

9) In the case where the two rotation circles about $\vec{\omega}_A$ and $\vec{\omega}_B$ coincide (nearly coincide) the three segment slew can be modified to obtain a faster slew. When the rotation circles about $\vec{\omega}_A$ and $\vec{\omega}_B$ coincide the motion is taking place in one plane and there is no need for plane changing. In this case, a more direct (and minimum time) trajectory is available. Specifically, instead of creating two rest points, a (much) faster trajectory can be obtained by the phase plane methods

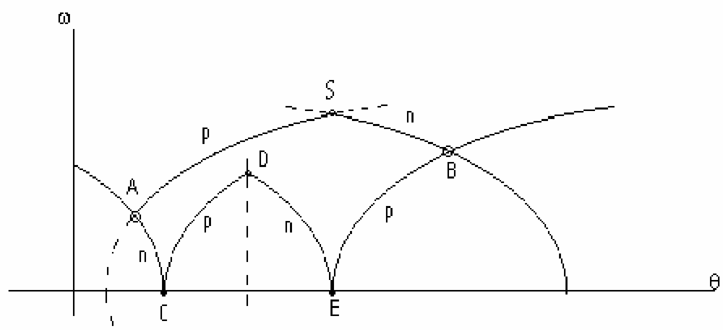


Figure 6: Single axis comparison of two slew procedures

implied in Figure 6 below. Slewing from *A* to *B* via *S* (positive acceleration then negative acceleration) is faster than the TSS algorithmic path *A-C-D-E-B*. Note, however, that the algorithm method can coincide with the minimum time bang-bang solution if certain points (positions and rates) align properly. For example, if points *A* and *B* are on the ω=0 axis (rest points) the algorithm coincides with the minimum time rest-to-rest solution. The point being made here is that the algorithm performs the required quaternion based slew (with non-resting endpoints), and that the procedure reduces to the standard rest-to rest slew.

Keep Out Regions:

Slewing issues are not restricted to finding compact motions between targets. It may be necessary to avoid direct paths between targets because of concern for sun light reflections from large bodies of water or flat and bright desert regions. In addition, housekeeping and calibration requirements will also require the spacecraft to slew to LOS directions above/beyond the Earth's horizon; i.e., celestial targets for aligning and calibrating star trackers, moon's surface for intensity calibration, and, perhaps, controlled rotations for gyro calibration. In any case, some otherwise suitable slewing paths may not be acceptable in order to avoid Keep Out (KO) zones in which the sensors may be damaged.

KO zones can be readily avoided by adding an avoidance feature to the TSS algorithm in a natural and simple manner. Figure 7 shows a TSS trajectory with an overlay of a KO zone. The KO zone is shown to interfere only with segment2 of the TSS because it is unlikely that it would interfere with either the first or third segment. To avoid the KO zone we create a trajectory arc on the unit sphere surface that misses/bypasses the zone to be avoided.

REMARK: If the KO zone were to interfere with the first segment, then the completed target would, almost by definition, be too close to the KO zone to have been selected in the first place. The end of segment1 is the "shortest braking distance" that the LOS could achieve and, thus, if the LOS enters the KO zone, the encounter/mission algorithm is bad and shouldn't have selected that target to begin with. The same remark applies to segment3.
ENDofREMARK:

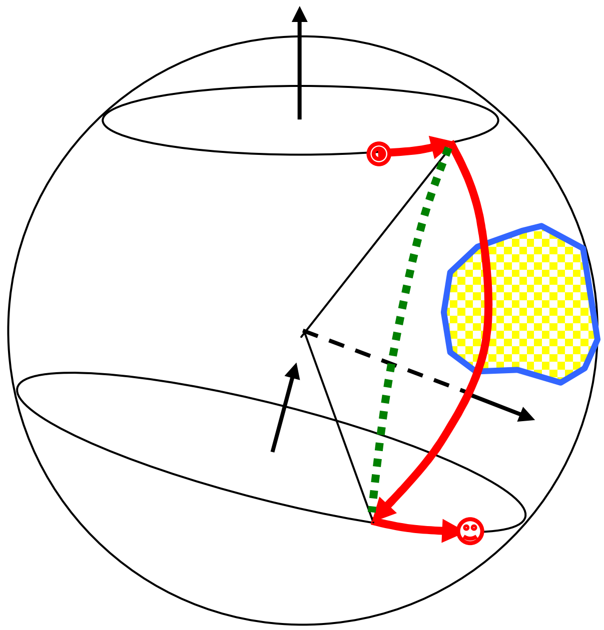


Figure 7: Keep Out zone

Construction of the avoidance arc will now be described. The rotation performed by segment2 is a rest to rest rotation and the paper by Reynolds[7] shows how to select rotation axes such that an initial direction is rotated into the final direction by various routes.

In Figure 8 the LOS direction $\vec{X}_A$ is to be rotated into direction $\vec{X}_B$. The natural solution is to determine the unit rotation axis $\hat{r}_1 = \left(\left(\vec{X}_A \times \vec{X}_B\right) / \left|\left(\vec{X}_A \times \vec{X}_B\right)\right|\right)$ and rotate the body about this axis until the angular amount $\theta_{end} = \arccos\left(\left(\vec{X}_A \bullet \vec{X}_B\right)\right)$ is achieved. The time varying rotation angle $\theta(t)$ depends on the control system capabilities and results in the time varying quaternion

$$\vec{Q}_1(t) = \left(\hat{r}_1 \sin\left(\theta(t)/2\right), \cos\left(\theta(t)/2\right)\right) \quad \text{where} \quad 0 \le \theta(t) \le \theta_{end}$$

and where $\theta(t) = F_1(t)$ is the function defining the evolution of θ as determined by the capabilities and constraints on the system. The attitude quaternion of the rigid body is the combination of the starting quaternion at the beginning of segment2, $\vec{Q}_S$, with $\vec{Q}_1$ to give the attitude quaternion of the motion for this segment

$$\vec{Q}(t) = \vec{Q}_S \otimes \vec{Q}_1(t).$$

Note that the rotation angle determined by this method is the smallest rotation angle required to rotate $\vec{X}_A$ into $\vec{X}_B$. The body rate during this segment is also required and given by

$$\vec{\omega} = \dot{\theta}(t)\hat{r}_1$$

since the only rotation is about the axis $\hat{r}_1$ in this segment2.

If the KO zone interferes with this direct (great circle) rotation, then an immediate alternative is to use the same rotation axis but rotate in the opposite direction. Thus,

$$\vec{Q}_{1alt}(t) = \left(-\hat{r}_1 \sin(\phi(t)/2), \cos(\phi(t)/2)\right) \quad \text{where, now} \quad 0 \geq \phi(t) \geq (\theta_{end} - 360)$$

As can be seen, this would require a very large rotation angle. For example, if $\theta = 30^o$, then the (easy) alternate route would require a rotation of 330^o and the added time to complete.

Another method of rotating $\vec{X}_A$ into $\vec{X}_B$ is to use the axis $\hat{r}_2 = \left((\vec{X}_A + \vec{X}_B)/|(\vec{X}_A + \vec{X}_B)|\right)$ which lies in the $(\vec{X}_A, \vec{X}_B)$ plane and bisects the angle between $\vec{X}_A$ and $\vec{X}_B$. A total rotation about this axis of 180 degrees will move $\vec{X}_A$ into $\vec{X}_B$. The intermediate rotation attitudes are determined by the quaternion

$$\vec{Q}_2(t) = \left(\pm \hat{r}_2 \sin(\theta(t)/2), \cos(\theta(t)/2)\right) \quad \text{where } 0 \leq \theta(t) \leq 180$$

and the sign in front of $\hat{r}_2$ depends on whether the upper or lower path is taken. As mentioned above, the time varying rotation angle $\theta(t)$ depends on the control system capabilities and has the form $\theta(t) = F_1(t)$ which defines the evolution of θ as determined by the capabilities and constraints on the system. The attitude quaternion of the rigid body is the combination of the

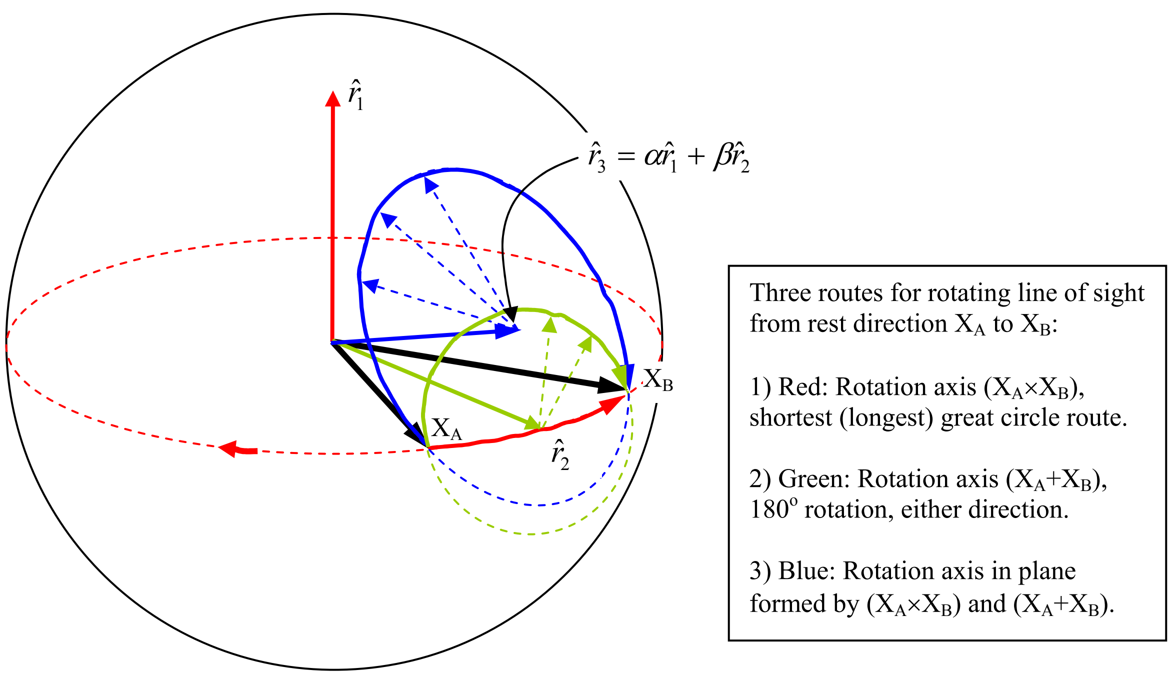


Figure 8: Avoiding keep out zones by selecting proper rotation axis. Adapted from article by Reynolds.

starting quaternion at the beginning of segment2, $\vec{Q}_S$, with $\vec{Q}_2$ to give the attitude quaternion of the motion for this segment

$$\vec{Q}(t) = \vec{Q}_S \otimes \vec{Q}_2(t)\,.$$

Note that a total rotation angle of 180 degrees about the axis $\hat{r}_2$ is required to rotate $\vec{X}_A$ into $\vec{X}_B$. In addition, this second rotation method will cause the field of view at the end of the LOS to be flipped over completely. The body rate during this segment is also required and given by exactly the same calculation provided immediately above, by

$$\vec{\omega} = \dot{\theta}(t)\hat{r}_2$$

since the only rotation is about the axis $\hat{r}_1$ in this segment2.

Finally, any unit rotation axis, $\hat{r}_3 = (\alpha\hat{r}_1 + \beta\hat{r}_2)/|(\alpha\hat{r}_1 + \beta\hat{r}_2)|$, lying in the plane formed by $\hat{r}_1$ and $\hat{r}_2$, will also rotate $\vec{X}_A$ into $\vec{X}_B$ as illustrated by the blue curve in Figure Y. The range of the angle of rotation $\theta_{3\max}$ to move from $\vec{X}_A$ into $\vec{X}_B$ is determined from the geometry of the situation. Thus,

$$\vec{Q}_3(t) = \left(\pm\,\hat{r}_3 \sin(\theta(t)/2), \cos(\theta(t)/2)\right) \ \text{ where } 0 \le \theta(t) \le \theta_{3\max}\,.$$

The same comments about the body attitude and rate mentioned above apply to this case as well.

When the TSS algorithm was presented the rotation method for segment2 was not specified. What was important was that the end points were achieved and were at rest. In most cases the direct route using $(\vec{X}_A \times \vec{X}_B)$ is used. But the alternate methods mentioned above fit into the TSS algorithm perfectly and the method illustrated in Figure 8 can be used to avoid KO zones.

The above discussion was based on the need to move only the LOS from one rest location to another. In the slew cases where a quaternion (and not merely LOS direction) match at each rest point is needed to define the segment2 rotation, the above procedure needs to be modified. At the first rest point the quaternion determines the LOS. We can also determine the direction of the LOS at the second rest point and then perform one of the rotations mentioned above to get to this second rest point. All that is being matched at the end of this procedure at the second rest point is the LOS directions; but the quaternion may not match since it depends on the route chosen. To achieve the correct quaternion attitude at the second rest point, we just perform the necessary rotation about the LOS axis. This, of course, adds time to the slew, but we avoid the KO zone. The KO zone avoidance maneuver combines naturally with the TSS algorithm.

Conclusion and Summary:

We have carefully defined the slewing problem and pointed out that it includes requirements and conditions not normally expected from a casual consideration of the issue. We have then pointed out the importance of rapid slewing and that achieving this is very difficult. A plausible solution approach that is less than optimal has been developed and described. This approach, the Three Segment Slew (TSS), is easy to implement, versatile, and adaptable to new and varied requirements. The TSS can also form the basis for future research into improved solutions if reasonable iterative schemes can be implemented.

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Appendix A: Illustration of the Implementation of Quaternion Three Segment Slew:

This appendix provides a detailed procedure for determining a version of the Three Segment Slew described in the body of this paper. Here it is assumed that the only constraints are limits on the maximum angular acceleration as determined by the maximum torque capability of the spacecraft, and body rate limit as would be determined by maximum reaction wheel speeds. For other constraint, such as on higher order motion derivatives (i.e., jerk limits to inhibit possible structural vibrations) a similar procedure can be followed, but is not considered here.

Given:
Initial Quaternion Phase Point $A = \{\vec{Q}_A, \vec{\omega}_A\}$, Final Quaternion Phase Point $B = \{\vec{Q}_B, \vec{\omega}_B\}$,
Body angular rate and acceleration limits, $\omega_{limit}, a_{\max}$

Determine: *TSlew*, the time required to slew from *A* to *B* using the Three Segment Slew procedure and incorporating system constraints and limitations. Equations used to determine the actual attitude quaternion and body rate, $(\vec{Q}(t), \vec{\omega}(t))$, for any time instant (or instants) during the slew period. Makes available to all necessary intermediate parameters.

PROCEDURE: Uses the TSS trajectory described earlier.

A-1.1) First slew segment, t_1 : start with the given initial quaternion phase point $\vec{Q}_A, \vec{\omega}_A$:
Align the first segment axis, $\vec{e}_1$, with $\vec{\omega}_A$ as follows:
Calculate

(1) $$W_1 = |\vec{\omega}_A| = +\sqrt{\omega_{A1}^2 + \omega_{A2}^2 + \omega_{A3}^2}$$: magnitude of angular velocity $\vec{\omega}_A$

Calculate

(2) $$\hat{e}_1 = [e_{11} \quad e_{12} \quad e_{13}]^T = \frac{\vec{\omega}_A}{W_1}$$:rotation axis (unit vector) in the $\vec{\omega}_A$ direction

The rotation acceleration vector, $\vec{\alpha}_1(t) = \dot{\vec{\omega}}_1(t)$, for this axis rotation, remains aligned with the rotation axis, $\hat{e}_1$. The angular acceleration available along axis $\hat{e}_1$ is

(3) $$\alpha_1 = Acc(\hat{e}_1)$$

The scalar function, $Acc(\hat{e})$, depends on spacecraft mass properties and torquer arrangement. For illustrative purposes here this can be considered a constant and equal to the acceleration limits, $\pm a_{\max}$.

The time required to reduce the body rate to zero using the maximum (negative) acceleration is

(4) $$t_1 = \left|\frac{W_1}{\alpha_1}\right|$$: time to reduce initial body rate to zero,

During this time the body rotates through the angle

(5) $$\theta_1 = \theta(t_1) = W_1 t_1 - \frac{1}{2}\alpha_1 t_1^2 = \frac{W_1^2}{2\alpha_1}$$: rotation angle to intermediate rest point, I_1

The rotation quaternion to the first rest point is

(6) $$\Delta\vec{q}_1 = \begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \\ q_{14} \end{bmatrix} = \begin{bmatrix} e_{11}\,sin(\tfrac{1}{2}\theta_1) \\ e_{12}\,sin(\tfrac{1}{2}\theta_1) \\ e_{13}\,sin(\tfrac{1}{2}\theta_1) \\ cos(\tfrac{1}{2}\theta_1) \end{bmatrix}$$: quaternion change to location $\vec{Q}_{I1}$

Locate $\vec{Q}_{I1}$, the first intermediate rest point $(\vec{Q}_{I1},\vec{0})$ by the following quaternion multiplication

(7) $$\vec{Q}_{I1} = \begin{bmatrix} q_{I11} \\ q_{I12} \\ q_{I13} \\ q_{I14} \end{bmatrix} = \begin{bmatrix} q_{14} & q_{13} & -q_{12} & q_{11} \\ -q_{13} & q_{14} & q_{11} & q_{12} \\ q_{12} & -q_{11} & q_{14} & q_{13} \\ -q_{11} & -q_{12} & -q_{13} & q_{14} \end{bmatrix} \begin{bmatrix} q_{A1} \\ q_{A2} \\ q_{A3} \\ q_{A4} \end{bmatrix}$$

The first rest point is thus $\{\vec{Q}_{I1},\vec{0}\}$.

A-1.2) Third slew segment, *t₃* : (second segment is considered last) begin with the required final quaternion phase point $\{\vec{Q}_B,\vec{\omega}_B\}$:
The third segment axis, $\vec{e}_3$, is aligned with $\vec{\omega}_B$ and determined as follows:
Calculate

(8) $$W_3 = |\vec{\omega}_B| = +\sqrt{\omega_{B1}^2 + \omega_{B2}^2 + \omega_{B3}^2}$$: magnitude of angular velocity $\vec{\omega}_B$

Calculate

(9) $$\hat{e}_3 = [e_{31} \quad e_{32} \quad e_{33}]^T = \frac{\vec{\omega}_B}{W_3}$$: axis (unit vector) in the $\vec{\omega}_B$ direction

The rotation acceleration vector, $\vec{\alpha}_3(t) = \dot{\vec{\omega}}_3(t)$, for this axis rotation, is aligned with the rotation axis, $\hat{e}_3$. The angular acceleration available along the axis $\hat{e}_3$ is

(10) $$\alpha_3 = Acc(\vec{e}_3)$$

As before, the scalar function $Acc(\hat{e})$ depends on the spacecraft's mass and torquing properties, and for purposes here is considered to be $\pm a_{\max}$.
This segment starts at the second rest point $\vec{Q}_{I2}$ (to be determined here) and accelerates and rotates as hard as possible to the endpoint $\{\vec{Q}_B,\vec{\omega}_B\}$.
The time required to increase the body rate from zero to $\vec{\omega}_B$ using the maximum acceleration is

(11) $$t_3 = \left|\frac{W_3}{\alpha_3}\right|$$: time to accelerate from rest to velocity W_3

During this phase the body rotate about the axis, $\hat{e}_3$, through the angle

(12) $$\theta_3 = \theta_3(t_3) = \frac{1}{2}\alpha_3 t_3^2 = \frac{W_3^2}{2\alpha_3}$$: rotation angle from rest to angular rate W_3.

Since we know the end quaternion, $\vec{Q}_B$, the starting attitude, $\vec{Q}_{I2}$, of this rotation is determined by the reverse of the rotation defined by $\hat{e}_3$ and θ_3. The quaternion back from $\vec{Q}_B$ to $\vec{Q}_{I2}$ is given by

(13) $$\Delta\vec{q}_3 = \begin{bmatrix} q_{31} \\ q_{32} \\ q_{33} \\ q_{34} \end{bmatrix} = \begin{bmatrix} e_{31}\, sin(-\frac{1}{2}\theta_3) \\ e_{32}\, sin(-\frac{1}{2}\theta_3) \\ e_{33}\, sin(-\frac{1}{2}\theta_3) \\ cos(-\frac{1}{2}\theta_3) \end{bmatrix}$$: quaternion change to location $\vec{Q}_B$.

This locates the second intermediate rest point $\vec{Q}_{I2}$ in $\{\vec{Q}_{I2}, \vec{0}\}$ as,

(14) $$\vec{Q}_{I2} = \begin{bmatrix} q_{I21} \\ q_{I22} \\ q_{I23} \\ q_{I24} \end{bmatrix} = \begin{bmatrix} q_{34} & q_{33} & -q_{32} & q_{31} \\ -q_{33} & q_{34} & q_{31} & q_{32} \\ q_{32} & -q_{31} & q_{34} & q_{33} \\ -q_{31} & -q_{32} & -q_{33} & q_{34} \end{bmatrix} \begin{bmatrix} q_{B1} \\ q_{B2} \\ q_{B3} \\ q_{B4} \end{bmatrix}$$

A-1.3) Second slew segment, t_2 : the connecting rest-to-rest slew, is determined as follows. *REMARK: It should be noted that during this rest-to-rest slew portion the body rate can (try to) exceed the system capabilities. Consequently, rate limits are imposed and they can affect the slew time for this portion.*

To go from $\vec{Q}_{I2}$ to $\vec{Q}_{I2}$ requires a slew quaternion given by

(15) $$\Delta\vec{q}_2 = \begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \\ q_{24} \end{bmatrix} = \begin{bmatrix} -q_{I24} & -q_{I23} & q_{I22} & q_{I21} \\ q_{I23} & -q_{I24} & -q_{I21} & q_{I22} \\ -q_{I22} & q_{I21} & -q_{I24} & q_{I23} \\ q_{I21} & q_{I22} & q_{I23} & q_{I24} \end{bmatrix} \begin{bmatrix} q_{I11} \\ q_{I12} \\ q_{I13} \\ q_{I14} \end{bmatrix} \Delta\vec{q}_2$$

From $\Delta\vec{q}_2$ determine

(16) $$\theta_2 = 2\,cos^{-1}(q_{24})$$: rotation angle for rest to rest slew

(17) $$\hat{e}_2 = \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} = \frac{1}{\sin\left(\frac{1}{2}\theta_2\right)} \begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \end{bmatrix}$$: normalized direction (eigenaxis) of segment2 rest-to-rest rotation

The rotation acceleration vector $\vec{\alpha}_2 = \dot{\vec{\omega}}_2$, for this eigenaxis rotation, is aligned with the rotation eigenaxis, $\hat{e}_2$. As before, the angular acceleration available along the eigenaxis $\hat{e}_2$ is

(18) $\alpha_2 = Acc(\vec{e}_2)$: scalar function $Acc(\vec{e})$ defined in section

With knowledge of the rotation axis, maximum accelerations, and body rates, it is possible to determine the time to complete segment2 of the slew. The basic calculation to obtain a (near) minimum time slew is to perform a bang-bang acceleration rotation about the eigenaxis. This is next described.

REMARK: If there were no rate limits on the system then the slew time for this portion is calculated by

$$t_2 = 2\sqrt{\frac{\frac{1}{2}\theta_2}{\frac{1}{2}\alpha_2}} = 2\sqrt{\frac{\theta_2}{\alpha_2}} \quad .$$

However, because of rate limits the following procedure is used instead.

First calculate the (unrestricted maximum slewing) angular velocity

(19) $\omega_{ums} = \sqrt{\alpha_2\theta_2}$

Calculate

(20) $\omega_D = min(\omega_{ums}, \omega_{limit})$

where ω_{limit} is the maximum allowable body rate in direction $\vec{e}_2$.

The slew time of this second segment is itself broken into three sub segments t_{21}, t_{22}, and t_{23} during which there is acceleration, coasting, and deceleration, respectively. The timing for these three sub segments is given by

(21a,b,c)

$$t_{21} = \frac{\omega_D}{\alpha_2}$$

$$t_{22} = \frac{\omega_{ums}^2 - \omega_D^2}{\alpha_2\omega_D} \quad : \text{ Note that } t_{22} = 0 \text{ if } \omega_{ums} < \omega_{limit}$$

$$t_{23} = \frac{\omega_D}{\alpha_2} = t_{21}$$

The time for segment 2 is

(22) $t_2 = t_{21} + t_{22} + t_{23} = 2t_{12} + t_{22}$

NOTE: if $\omega_{ums} < \omega_{limit}$, then the above sum reduces to $t_2 = 2\sqrt{\frac{\theta_2}{\alpha_2}}$.

Finally, the total time for the slew is

(23) $TSlew = t_1 + t_2 + t_3$

The above procedure gives the timing, and as a byproduct, the few parameters necessary to reproduce the trajectory at any time step.

Determining Slew attitude quaternion and rate for any time during slew period:

The parameters from above that allow the calculation of the slew attitude quaternion and rate are:

$\alpha_1,\ W_1,\ t_1,\ \vec{e}_1$

$\alpha_2,\ t_2,\ t_{21},\ t_{22},\ \vec{e}_2, \vec{Q}_{I1}$

$\alpha_3,\ t_3,\ \vec{e}_3, \vec{Q}_{I2}$

This following additional steps are required to complete the trajectory calculation.

At this point we have:

Initial Quaternion Phase Point A = $(\vec{Q}_A, \vec{\omega}_A)$,

Final Quaternion Phase Point B $= (\vec{Q}_B, \vec{\omega}_B)$

Body rate limit, ω_{limit}

Calculate: A quaternion trajectory that slews from *A* to *B* and uses body rate limits and acceleration capabilities of the system.

<u>PROCEDURE:</u> Uses the Three Segment Slew trajectory described earlier.
With input $\vec{Q}_A, \vec{\omega}_A$, and the remaining parameters, proceed as follows:

<u>First slew segment:</u> time interval $(0 < t < t_1)$ the trajectory quaternion is determined using the following steps

(24) $$\theta_1(t) = W_1 t - \frac{1}{2}\alpha_1 t^2 \qquad : 0 < t < t_1$$

(25) $$\Delta\vec{q}_1(t) = \begin{bmatrix} q_{11}(t) \\ q_{12}(t) \\ q_{13}(t) \\ q_{14}(t) \end{bmatrix} = \begin{bmatrix} e_{11}\, sin(\frac{1}{2}\theta_1(t)) \\ e_{12}\, sin(\frac{1}{2}\theta_1(t)) \\ e_{13}\, sin(\frac{1}{2}\theta_1(t)) \\ cos(\frac{1}{2}\theta_1(t)) \end{bmatrix} \qquad : \text{quaternion change for } 0 < t < t_1$$

(26) $$\vec{Q}(t) = \begin{bmatrix} q_{14}(t) & q_{13}(t) & -q_{12}(t) & q_{11}(t) \\ -q_{13}(t) & q_{14}(t) & q_{11}(t) & q_{12}(t) \\ q_{12}(t) & -q_{11}(t) & q_{14}(t) & q_{13}(t) \\ -q_{11}(t) & -q_{12}(t) & -q_{13}(t) & q_{14}(t) \end{bmatrix} \begin{bmatrix} q_{A1} \\ q_{A2} \\ q_{A3} \\ q_{A4} \end{bmatrix} \qquad : 0 < t < t_1$$

The body rate during this interval is determined from

(27) $$\vec{\omega}(t) = \vec{\omega}_A - \alpha_1 t \vec{e}_1 = (W_1 - \alpha_1 t)\vec{e}_1 \qquad : 0 < t < t_1$$

<u>Second slew segment:</u> time interval is (t_1, t_2) but is subdivided into three segments of duration t_{21}, t_{22}, t_{23}. The quaternion trajectory and the body rate in these three regions is given by

For time interval: $t_1 < t < t_1 + t_{21}$

(28) $\theta_2(t) = \frac{1}{2}\alpha_2(t - t_1)^2$: rotation angle along e_2 eigenaxis

(29) $Q(t)$ = (use equations (37) and (38) below)

(30) $\vec{\omega}(t) = (a_2(t - t_1))\vec{e}_2$

For time interval: $t_1 + t_{21} < t < t_1 + t_{21} + t_{22}$

(31) $\theta_2(t) = \frac{1}{2}\alpha_2 t_{21}^2 + (\alpha_2 t_{21})t$: rotation angle along e_2 eigenaxis

(32) $Q(t)$ = (use equations (37) and (38) below)

(33) $\vec{\omega}(t) = (\alpha_2 t_{21})\vec{e}_2$

For time interval: $(t_1 + t_{21} + t_{22}) < t < (t_1 + t_2)$

(34) $\theta_2(t) = \frac{1}{2}\alpha_2 t_{21}^2 + \alpha_2 t_{21} t_{22} + \alpha_2 t_{21}(t - (t_1 + t_{21} + t_{22})) - \frac{1}{2}\alpha_2(t_1 + t_{21} + t_{22})^2$

(35) Q(t) = (use equations (15) and (16) below)

(36) $\vec{\omega}(t) = (\alpha_2 t_{21} - \alpha_2(t - (t_1 + t_{21} + t_{22}))\vec{e}_2$

(37) $$\Delta\vec{q}_2(t) = \begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \\ q_{24} \end{bmatrix} = \begin{bmatrix} e_{21}\sin(\frac{1}{2}\theta_2(t)) \\ e_{22}\sin(\frac{1}{2}\theta_2(t)) \\ e_{23}\sin(\frac{1}{2}\theta_2(t)) \\ \cos(\frac{1}{2}\theta_2(t)) \end{bmatrix}$$; quaternion over the range $t_1 < t < t_1 + t_2$

(38) $$\vec{Q}(t) = \begin{bmatrix} q_{24}(t) & q_{23}(t) & -q_{22}(t) & q_{21}(t) \\ -q_{23}(t) & q_{24}(t) & q_{21}(t) & q_{22}(t) \\ q_{22}(t) & -q_{21}(t) & q_{24}(t) & q_{23}(t) \\ -q_{21}(t) & -q_{22}(t) & -q_{23}(t) & q_{24}(t) \end{bmatrix} \begin{bmatrix} q_{I11} \\ q_{I12} \\ q_{I13} \\ q_{I14} \end{bmatrix}$$

<u>**Third slew segment**</u>: time interval is $(t_2 < t < t_3)$

(39) $\theta_3(t) = \frac{1}{2}\alpha_3(t - (t_2 + t_3))^2$

(40) $$\Delta\vec{q}_3(t) = \begin{bmatrix} q_{31}(t) \\ q_{32}(t) \\ q_{33}(t) \\ q_{34}(t) \end{bmatrix} = \begin{bmatrix} e_{31}\sin(\frac{1}{2}\theta_3(t)) \\ e_{32}\sin(\frac{1}{2}\theta_3(t)) \\ e_{33}\sin(\frac{1}{2}\theta_3(t)) \\ \cos(\frac{1}{2}\theta_3(t)) \end{bmatrix}$$: quaternion change for $t_2 < t < t_2 + t_3$

(41) $$\vec{Q}(t)=\begin{bmatrix} q_{34}(t) & q_{33}(t) & -q_{32}(t) & q_{31}(t) \\ -q_{33}(t) & q_{34}(t) & q_{31}(t) & q_{32}(t) \\ q_{32}(t) & -q_{31}(t) & q_{34}(t) & q_{33}(t) \\ -q_{31}(t) & -q_{32}(t) & -q_{33}(t) & q_{34}(t) \end{bmatrix} \begin{bmatrix} q_{I21} \\ q_{I22} \\ q_{I23} \\ q_{I24} \end{bmatrix}$$

The body rate over this interval is given by

(42) $$\vec{\omega}(t)=(\alpha_3(t-t_2))\vec{e}_3$$

END OF CALCULATION: